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A NECESSARY AND SUFFICIENT CONDITION FOR THE EXISTENCE OF A STIELTJES INTEGRAL

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The purpose of this note is to suggest a proof of the following theorem: *If* $u(x)$ *is of limited variation and* $f(x)$ *bounded on the interval* $a \leqq x \leqq b$, *then a necessary and sufficient condition for the existence of the Stieltjes integral*

$$\int_a^b f du \tag{1}$$

is that the total variation of $u(x)$ *on the set* D *of discontinuities of* $f(x)$ *be zero. If* $U(x)$ *is the total variation of* $u(x)$ *on the interval* ax, *and* α *an interval with end points* x_1, x_2, *then by definition*

$$U(\alpha) = U(x_2) - U(x_1)$$

and the total variation of $u(x)$ *on* D *is defined to be the greatest lower bound of the sums*

$$\sum_{k=1}^{\infty} U(\alpha_k)$$

for denumerable sequences of intervals $\{\alpha_k\}$ *containing the points of* D *as interior points of the set which the intervals define.*

Let the interval ab be subdivided by points x_i $(i = 0, 1, \ldots, n)$ with $x_0 = a$, $x_n = b$, $0 < x_i - x_{i-1} < \delta$ $(i = 1, 2, \ldots, n)$, and let S denote the sum

$$S = \sum_{i=1}^{n} f(X_i)\,[u(x_i) - u(x_{i-1})] = \sum_{i=1}^{n} f(X_i)\,\Delta_i\, u,$$

where the values X_i are arbitrarily selected in the intervals $x_{i-1}x_i$.

Then a necessary condition that the sums S have a limit as δ approaches zero is that

$$\lim_{\delta=0} \sum_{i=1}^{n} O_i \,|\, \Delta_i u \,| = 0, \tag{2}$$

where O_i is the oscillation of $f(x)$ on the interval $x_{i-1}x_i$. For it is clear that the values X_i can always be selected for two sums S, S' on the same partition so that the difference $S - S'$ is as near as is desired to the sum in (2). Hence if for every norm δ a partition can be found with

$$\sum_{i=1}^{n} O_i \,|\, \Delta_i u \,| > \eta > 0,$$

then for every δ two sums S, S' can be found with $|\, S - S' \,| > \eta$, and this would contradict the existence of the limit (1).

Let D_ϵ denote the closed set of points x where the oscillation $O(x)$ of f satisfies the inequality $O(x) \geqq \epsilon$, and let the intervals of S which contain points of D_ϵ as interior points be designated by the subscript j. Then it is also necessary that

$$\lim_{\delta=0} \sum_{j} |\, \Delta_j u \,| = 0. \tag{3}$$

For in every such interval $O_j \geqq \epsilon$, and if for every δ a sum as in (3) exists exceeding η, a sum as in (2) can be found exceeding $\epsilon\eta$.

Consider now a function $u(x)$ which is continuous as well as of limited variation, and suppose that the integral (1) exists. Then if δ is sufficiently small the j-intervals of S will be such that

$$\sum_{j} |\, \Delta_j u \,| < \eta.$$

Since $u(x)$ is now continuous the sum S can be altered by the introduction of new division points so that all the points of D are interior to the set of points defined by the sum of the j-intervals and the last inequality still true. Furthermore for the altered S it will follow that

$$\sum_{j} \Delta_j U < 2\eta$$

provided that δ is chosen so small that the total variation $U(b) - U(a)$, and $\sum_{i=1}^{n} |\, \Delta_i u \,|$, and hence also the corresponding quantities for every sum of partial intervals of ab, differ by less than η. Such a choice is always possible when $u(x)$ is continuous (Vallée Poussin, *Cours d'Analyse Infinitésimale*, vol. 1, 3d ed., p. 73). It follows then that the total variation of u on the set D_ϵ must be zero, and hence also the total variation of u on the totality D of discontinuities of $f(x)$, since D is the sum of the sets $D_{1/n}$ $(n = 1, 2, \dots)$.

To prove the sufficiency of the condition of the theorem for a function $u(x)$ which is continuous, consider the two monotonically increasing continuous functions $P(x)$, $N(x)$ satisfying the relations (Op. cit., p. 72)

$$u(x) - u(a) = P(x) - N(x), \; U(x) = P(x) + N(x). \qquad (4)$$

If the variation of $U(x)$ is zero on the set D then the same is clearly true of $P(x)$ and $N(x)$. Hence it is only necessary to prove the sufficiency for a monotonically increasing function $P(x)$. If D can be enclosed in the interior of a denumerably infinite system of intervals on which the sum of the variations of $P(x)$ is less than η the same is clearly true of its sub-set D_ϵ, and the latter is furthermore entirely interior to a finite number of the intervals since it is closed. (This follows from the Heine-Borel theorem. See, for example, Op. cit., p. 60, Lemme I. The proof is quite similar for a closed set or an interval.) The portion of ab remaining can be subdivided into intervals so small after D_ϵ is extracted that on each of them the oscillation of $f(x)$ is less than 2ϵ since on each such portion $O(x) < \epsilon$ (Op. cit., p. 254, §242). For every η and ϵ a partition of ab is defined in this way for which the inequality

$$\sum_{i=1}^{n} O_i \Delta_i P \leqq 2M\eta + 2\epsilon[P(b) - P(a)]$$

is true, where M is the upper bound of $|f|$ on ab. The first term on the right dominates the part of the sum for which the intervals contain points of D_ϵ, while the second does the same for the remaining terms. Hence the lower bound of the sum on the left is zero. But this is a sufficient condition that the integral

$$\int_a^b f\,dP$$

exists, as may be proved by precisely the method usually applied to the integral of Riemann (Op. cit., pp. 250, 255). It follows with the help of (4) that the integral (1) exists since the same is true when u is replaced by either P or N, and $u(x) - u(a) = P - N$.

The only case remaining to be considered is that of a function $u(x)$ of limited variation which has discontinuities. It is known that the totality of discontinuities of such a function is denumerable, and $u(x)$

is expressible as a sum $u(x) = v(x) + j(x)$, where $v(x)$ is of limited variation and continuous, and $j(x)$ is the 'function of jumps'

$$j(x) = \sum_{a<\xi\leqq x} [u(\xi) - u(\xi - 0)] + \sum_{a\leqq\xi<x} [u(\xi + 0) - u(\xi)]. \tag{5}$$

The sums on the right are taken in each case for all the discontinuities ξ satisfying the inequality below the summation sign, and are absolutely convergent.

It can be shown first of all that if the integral (1) exists the discontinuities of f and u are necessarily distinct. For suppose there were a discontinuity ξ common to both with $|u(\xi + 0) - u(\xi - 0)| > \eta$, $O(\xi) > \eta$. On a sufficiently small interval $\xi - h \leqq x \leqq \xi + h$ the absolute value of $u(\xi + h) - u(\xi - h)$ and the oscillation of f would both exceed η. Hence for every norm δ there would exist two sums S and S' coinciding except on this interval and such that $|S - S'| > \eta^2$, which contradicts the existence of the limit (1). If $u(\xi - 0) = u(\xi + 0)$ at a discontinuity ξ, sums contradicting the existence of the limit can be similarly constructed by using intervals having ξ as end point.

In the second place the integral (1) with $j(x)$ in place of $u(x)$ always exists if the discontinuities of f and u are distinct, and it has the value

$$\int_a^b f\,dj = \sum_{\xi} f(\xi)\,[u(\xi+0) - u(\xi - 0)], \tag{6}$$

the sum being taken for all discontinuities ξ of $u(x)$ with the understanding that $u(a - 0) = u(a)$, $u(b + 0) = u(b)$. The series (5) for $x = b$ converge absolutely, and it is possible to select a set of discontinuities ξ_k $(k = 1, 2, \ldots, m)$ so that the sum of the absolute values of the terms of (5) not involving the ξ_k's is less than ϵ, and the corresponding sum from (6) less than ϵM. Consider now a sum S for the integral (6) with norm δ so small that no two values ξ_k can lie in the same interval. The value of S is the sum of the terms of $j(b)$ from (5) each multiplied by a factor of the form $f(X)$ with $|X - \xi| < \delta$. The terms not involving the ξ_k's have a sum less than ϵM in absolute value, while those involving the ξ_k's will differ from the corresponding terms of (6) by less than ϵM, say, if δ is sufficiently small since f is continuous at every ξ. Hence for a sufficiently small δ, the sum S differs from the sum on the right in (6) by less than $3\,\epsilon M$, which proves the statement at the beginning of this paragraph.

It follows then that a necessary and sufficient condition for the integral (1) to exist is that f and u have no discontinuity in common, and that the integral shall exist with v in place of u. For since $u = v + j$ every sum S for u is the sum of two similar ones for v and j, and when f and u have no common discontinuity the last has a definite limit as δ approaches zero, as has been shown. This condition is, however, equivalent to that of the theorem. For the total variation of u on an interval is the sum of the total variations of v and j, and the fact that the set of points common to two denumerable sets of intervals is itself representable as such a denumerable set, makes it possible to show that the total variation of u on a set of points D is also the sum of the similar variations for v and j. Furthermore the total variation $J(x)$ of j is the value obtained from the series (5) by replacing each term by its absolute value, and on a set of points D containing no discontinuity of u it is zero. For if the ξ_k's are a set of points as described above, a denumerable set A of non-overlapping intervals can be selected approaching each ξ_k as a limit on both sides and containing all the points of ab except the ξ_k's. This set of intervals will also enclose D in its interior since D contains no ξ_k, and the sum of the total variations of j on A will surely be less than ϵ. It is easy to see, conversely, that when the variation of j is zero on D then the latter contains no discontinuity of u. Hence the existence of the integral with v in place of u and the conditions that f and u have no discontinuity in common imply that the total variation of u on D is zero, and conversely, which was to be proved.

TRANSFORMATIONS OF APPLICABLE CONJUGATE NETS OF CURVES ON SURFACES

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When the rectangular point coördinates x, y, z of a surface satisfy an equation of the form

$$\frac{\partial^2\theta}{\partial u\partial v} = a\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v}, \tag{1}$$

the curves $u=$const., $v=$const. form a conjugate system. We assume that the parametric system of curves is of this sort throughout this note, and we shall speak of it as a *net*. Equation (1) is the *point equation* of the net.

If N is such a net, the coördinates x', y', z' of a second net N' are given by quadratures of the form